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THE ASYMPTOTIC DISTRIBUTION OF CERTAIN CHARACTERISTIC ROOTS AND VECTORS

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1. Introduction

In a number of problems in multivariate statistical analysis use is made of characteristic roots and vectors of one sample covariance matrix in the metric of another. If A^* and D^* are the sample matrices, we are interested in the roots ϕ^* of $|D^* - \phi^* A^*| = 0$ and the associated vectors satisfying $D^* c^* = \phi^* A^* c^*$. In the cases we consider A^* and D^* have independent distributions. Each is distributed like a sum $\sum_{\alpha=1}^{q} y_\alpha y'_\alpha$ where $y_1, \ldots, y_q$ are independently normally distributed with common covariance matrix. In the case of A^* the means of the vectors are zero; in the case of D^* the means may not be zero. We are interested in the asymptotic distribution of the characteristic roots and vectors when the number of vectors defining A^* increases indefinitely and when the means of the vectors defining D^* change in a certain way. The form of the limiting distribution depends on the multiplicity of the roots of a certain determinantal equation involving the parameters. If these roots are simple and different from zero, the asymptotic distribution is joint normal. If the roots are not simple, the asymptotic distribution is expressed in terms of "uniform distributions" on orthogonal matrices and a normal distribution.

We shall first state our problem in a general form and show in what kinds of statistical problems there is interest in these characteristic roots and vectors. Suppose[1] $\mathbf{x}_\alpha (\alpha = 1, \ldots, N)$ of p components is normally distributed independently of $\mathbf{x}_\beta (\alpha \neq \beta)$ with mean

$$\mathcal{E} \mathbf{x}_\alpha = \mathbf{B}_1 \mathbf{z}_{1\alpha} + \mathbf{B}_2 \mathbf{z}_{2\alpha} \tag{1.1}$$

and covariance

$$\mathcal{E} (\mathbf{x}_\alpha - \mathcal{E} \mathbf{x}_\alpha)(\mathbf{x}_\alpha - \mathcal{E} \mathbf{x}_\alpha)' = \mathbf{\Sigma}\,, \tag{1.2}$$

where $\mathbf{z}_{1\alpha}$ and $\mathbf{z}_{2\alpha}$ are vectors of fixed variates of q_1 and q_2 components, respectively, and $\mathbf{B}_1$ and $\mathbf{B}_2$ are $p \times q_1$ and $p \times q_2$ matrices, respectively. We shall use the notation $N(\mathbf{B}_1 \mathbf{z}_{1\alpha} + \mathbf{B}_2 \mathbf{z}_{2\alpha}, \mathbf{\Sigma})$ for the distribution of $\mathbf{x}_\alpha$.

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[1] Unless specifically indicated otherwise, a vector is a column vector; a prime indicates the transpose of a vector or matrix. Vectors and matrices are indicated by bold face type.

On the basis of a sample $(x_1, z_{11}, z_{21}), \ldots, (x_N, z_{1N}, z_{2N})$ the usual estimate of $\mathbf{B} = (\mathbf{B}_1\mathbf{B}_2)$ is

$$B = \sum_{\alpha=1}^{N} x_\alpha z'_\alpha \Big(\sum_{\alpha=1}^{N} z_\alpha z'_\alpha \Big)^{-1}, \tag{1.3}$$

where $z'_\alpha = (z'_{1\alpha}\, z'_{2\alpha})$ and $\sum_{\alpha=1}^{N} z_\alpha z'_\alpha$ is assumed to be nonsingular. The columns of B, say b_u, are normally distributed with means β_u, the corresponding columns of $\mathbf{B}$, and covariance

$$\mathcal{E}\,(b_u - \beta_u)\,(b_v - \beta_v)' = \Big(\sum_{\alpha=1}^{N} z_\alpha z'_\alpha \Big)^{-1}_{uv} \mathbf{\Sigma}\,, \tag{1.4}$$

where $\Big(\sum z_\alpha z'_\alpha \Big)^{-1}_{uv}$ indicates the element of the inverse matrix in the u-th row and v-th column.

Let Q^{-1} be the submatrix of $\Big(\sum z_\alpha z'_\alpha \Big)^{-1}$ consisting of the last q_2 rows and q_2 columns; this is also given by

$$Q = \sum_{\alpha=1}^{N} z_{2\alpha} z'_{2\alpha} - \sum_{\alpha=1}^{N} z_{2\alpha} z'_{1\alpha} \Big(\sum_{\alpha=1}^{N} z_{1\alpha} z'_{1\alpha} \Big)^{-1} \sum_{\alpha=1}^{N} z_{1\alpha} z'_{2\alpha}\,. \tag{1.5}$$

Then $(B_2 - \mathbf{B}_2)\,Q(B_2 - \mathbf{B}_2)'$ has a Wishart distribution with covariance matrix $\mathbf{\Sigma}$ and q_2 degrees of freedom, denoted by $W(\mathbf{\Sigma}, q_2)$. If $q_2 < p$, this distribution, called *singular*, is the distribution of $\sum_{u=1}^{q_2} y_u y'_u$ where y_u is distributed according to $N(0, \mathbf{\Sigma})$ independently of y_v $(u \neq v)$. The usual estimate of $\mathbf{\Sigma}$ is

$$A^* = \sum_{\alpha=1}^{N} (x_\alpha - B z_\alpha)\,(x_\alpha - B z_\alpha)' = \sum_{\alpha=1}^{N} x_\alpha x'_\alpha - B \sum_{\alpha=1}^{N} z_\alpha z'_\alpha B' \tag{1.6}$$

divided by $N - (q_1 + q_2)$. This matrix is distributed according to $W(\mathbf{\Sigma}, N - q_1 - q_2)$ independently of B.

Many statistical problems, for example [3], [6], involve the roots of

$$|B_2 Q B'_2 - \phi^* A^*| = 0\,, \tag{1.7}$$

or the vectors, for example [1], [2], satisfying

$$(B_2 Q B'_2 - \phi^* A^*)\, c^* = 0\,. \tag{1.8}$$

The p algebraically independent vectors c_g satisfying (1.8) may be normalized by

$$c^{*\prime}_g A^* c^*_h = n\,\delta_{gh}\,, \tag{1.9}$$

where $n = N - q_1 - q_2$ and $\delta_{gg} = 1$ and $\delta_{gh} = 0$ for $g \neq h$. We say that the solutions of (1.7) and (1.8) are the "characteristic roots and vectors of $B_2 Q B'_2$ in the metric of A^*." If we wish to test the hypothesis that the rank of $\mathbf{B}_2$ is r against the alternatives that it is greater than r we use the $p - r$ smallest roots of (1.7). If we assume that the rank of $\mathbf{B}_2$ is r and we wish to estimate $\mathbf{B}_2$ (or, equivalently,

estimate the linear restrictions on $\mathbf{B}_2$) we make use of the vectors $\mathbf{c}^*$ satisfying (1.8) for the $p - r$ smallest roots of (1.7).

In this paper we shall study the joint asymptotic distribution of the roots and vectors defined by (1.7), (1.8), and (1.9) when $n = N - q_1 - q_2 \to \infty$ and $\frac{1}{n}\sum_{\alpha=1}^{N} z_\alpha z_\alpha'$ approaches a nonsingular limit. The asymptotic distribution of the roots alone has been given by Hsu [8]. We find it convenient to make use of some of the results in [8] to obtain the joint asymptotic distribution of roots and vectors; however, the method used in the present paper could be used independently of [8]. We shall assume throughout the paper that $q_2 \geqq p$.

2. Reduction of the problem to canonical form

To simplify the following derivations we shall transform the matrices B_2QB_2' and A^* so that they have distributions with fewer parameters. Corresponding to (1.7) and (1.8) in the sample, we have the population equations

$$|B_2\bar{Q}_nB_2' - \tau^2\Sigma| = 0 \tag{2.1}$$

and

$$(B_2\bar{Q}_nB_2' - \tau^2\Sigma)\,\gamma = 0\ , \tag{2.2}$$

where $\bar{Q}_n = \frac{1}{n}Q$. Let the roots of (2.1) be $\tau_1^2(n) \geqq \tau_2^2(n) \geqq \ldots \geqq \tau_p^2(n) \geqq 0$. The number of zero roots is the difference of p and the rank of $\mathbf{B}_2$ for each n for which $\bar{Q}_n$ is nonsingular (in particular for n sufficiently large). Let $\gamma_1(n), \ldots, \gamma_p(n)$ be a set of corresponding solutions of (2.2) satisfying

$$\gamma_g'(n)\,\Sigma\gamma_h(n) = \delta_{gh}\ . \tag{2.3}$$

Let $\Gamma_n = [\gamma_1(n), \ldots, \gamma_p(n)]$. Then we can make a transformation, for example [7], so that A^* is replaced by

$$A_n = \sum_{\beta=1}^{n} y_\beta^* y_\beta^{*\prime}\ , \tag{2.4}$$

where y_β^* is distributed according to $N(\mathbf{0}, I)$ independently of y_α^* $(\beta \neq \alpha)$, and $B_2Q_nB_2'$ is replaced by

$$D_n = \sum_{g=1}^{q_2} y_g^{**}(n)\, y_g^{**\prime}(n) \tag{2.5}$$

where $y_g^{**}(n)$ is distributed independently of $y_h^{**}(n)$ $(g \neq h)$ according to $N[\sqrt{n}\tau_g(n)\varepsilon_g, I]$ where $\tau_g(n)$ is the nonnegative square root of $\tau_g^2(n)$ and ε_g is a vector with all components 0 except the g-th (for $g \leqq p$) which is 1. The roots of (1.7) are the roots of

$$|D_n - \phi A_n| = 0\ , \tag{2.6}$$

and the vectors satisfying (1.8) and (1.7) are related to the vectors $c_1(n), \ldots, c_p(n)$ satisfying

$$(D_n - \phi A_n^*)\, c = 0 \tag{2.7}$$

and

$$c_g' A_n c_h = n\,\delta_{gh} \tag{2.8}$$

by

$$c_g^*(n) = \Gamma_n c_g\,. \tag{2.9}$$

It should be observed that Γ_n and $\tau_i^2(n)$ depend on n because Q_n depends on n.

We shall first find the limiting distribution of $c_g(n)$ and $\phi_g(n)$ as $n \longrightarrow \infty$ (that is, as $N \longrightarrow \infty$). Let $y_g = y_g^{**}(n) - \sqrt{n}\tau_g(n)\varepsilon_g$. Then

$$D_n = \sum_{g=1}^{q_2} [y_g + \sqrt{n}\,\tau_g(n)\,\varepsilon_g][y_g + \sqrt{n}\,\tau_g(n)\,\varepsilon_g]'\,, \tag{2.10}$$

and y_g is distributed according to $N(0, I)$.

Let $C_n = [c_1(n), \ldots, c_p(n)]$. Then (2.7) can be written

$$D_n C_n^* = A_n C_n^* \Phi_n\,, \tag{2.11}$$

where $\Phi_n = [\phi_i(n)\delta_{ij}]$ and $\phi_1(n), \ldots, \phi_p(n)$ are the roots of (2.6) and (2.8) can be written

$$C_n' A_n C_n = nI\,. \tag{2.12}$$

If

$$X_n = C_n^{-1}\,, \tag{2.13}$$

we have

$$\frac{1}{n} A_n = X_n' X_n\,, \tag{2.14}$$

$$\frac{1}{n} D_n = X_n' \Phi_n X_n\,. \tag{2.15}$$

We shall set out to find the limiting distribution of Φ_n and X_n for $\tau_g(n)$ approaching limits as $n \longrightarrow \infty$. To make Φ_n and X_n unique we require $\phi_1(n) > \phi_2(n) > \ldots > \phi_p(n)$ and $x_{i1}(n) > 0$. The probability is 0 of a D_n and A_n for which X_n and Φ_n are not uniquely defined.

Throughout this paper we shall make use of the following special case of a theorem of Rubin [9]:

RUBIN'S THEOREM: *Let $F_n(u)$ be the cumulative distribution function of a random vector u_n. Let v_n be a (vector valued) function of u_n, $v_n = f_n(u_n)$, and let $G_n(v)$ be the (induced) distribution of v_n. Suppose $\lim_{n\to\infty} F_n(u) = F(u)$ [in every continuity point of $F(u)$] and suppose for every continuity point u of $f(u)$, $\lim_{n\to\infty} f_n(u_n) = f(u)$, when $\lim_{n\to\infty} u_n = u$. Let $G(v)$ be the distribution of the random vector $v = f(u)$, where u has the distribution $F(u)$. If the probability of the set of discontinuities of $f(u)$ in terms of $F(u)$ is 0, then*[2]

$$\lim_{n\longrightarrow\infty} G_n(v) = G(v)\,.$$

[2] We could justify the limiting procedures by another method that consists of extending a theorem of L. C. Young ("Limits of Stieltjes integrals," *Jour. London Math. Soc.*, Vol. 9 [1934], pp. 119–126), concerning the limit of $\int g_n(u)dF_n(u)$, applying this to the characteristic function of $f_n(u_n)$, and thus obtaining a restricted form of Rubin's theorem.

In our case the components of u_n are linear combinations of the components of the matrices A_n and D_n; the components of v_n are linear combinations of the characteristic roots and the components of the characteristic vectors. The distribution of u_n approaches a limit and the function $f_n(u)$ approaches a limit (in the above sense). We shall verify that the discontinuities of the limiting function are of limiting probability zero. Thus we can deduce the asymptotic distribution of the characteristic roots and (normalized) vectors by using the asymptotic distribution of A_n and D_n and the limiting function.

3. Derivation of two special distributions

In order to derive the desired asymptotic distributions we need to obtain the distributions of the characteristic roots and vectors (in the metric of I) of a symmetric matrix B in two special cases. Let the roots of

$$|B - \psi I| = 0 \tag{3.1}$$

be $\psi_1 \geqq \psi_2 \geqq \ldots \geqq \psi_p$. Let the characteristic vector satisfying

$$Bh = \psi_i h\,, \tag{3.2}$$

and $h'h = 1$ be h_i $(i = 1, \ldots, p)$. If $\psi_1, \ldots, \psi_p$ are different $h_1, \ldots, h_p$ are uniquely defined except for multiplication of a vector by -1, and $h_i'h_j = 0$, $i \neq j$. Let $H = (h_1, \ldots, h_p)$. Then $BH = H\Psi$, where $\Psi = (\psi_i \delta_{ij})$. Let $H' = G$. Then G satisfies

$$G'\Psi G = B\,, \tag{3.3}$$

$$G'G = I\,. \tag{3.4}$$

These equations define Ψ and G uniquely if we require $g_{i1} \geqq 0$ except for a set of B of measure zero. Since it is trivial to obtain the distribution of H and Ψ from that of G and Ψ, we shall now obtain the distribution of G and Ψ.

First we consider the case that the distribution of B is $W(I, m)$ $(m \geqq p)$; that is, the density is

$$C(m, p)\ |B|^{(m-p-1)/2} e^{-\operatorname{tr} B/2} \tag{3.5}$$

where

$$C^{-1}(m, p) = 2^{mp/2} \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left[\tfrac{1}{2}(m + 1 - i)\right] \tag{3.6}$$

and "tr" denotes trace. This is the distribution of

$$B = \sum_{f=1}^{m} u_f u_f'\,, \tag{3.7}$$

where $u_1, \ldots, u_m$ are independently distributed according to $N(0, I)$.

THEOREM 1. *Let B have the distribution $W(I, m)$. Then G and Ψ, defined by* (3.3), (3.4), *the restriction that Ψ is diagonal with diagonal elements in descending order*

and $g_{i1} \geqq 0$, *are independently distributed. The density of the diagonal elements of* Ψ *is*

$$(3.8)\qquad \frac{\pi^{p/2}}{2^{pm/2}\prod_{i=1}^{p}\{\Gamma[\frac{1}{2}(m+1-i)]\Gamma[\frac{1}{2}(p+1-i)]\}}\prod_{i=1}^{p}\psi_i^{(m-p-1)/2}e^{-\sum_{i=1}^{p}\psi_i/2} \times\prod_{i=1}^{p}\prod_{j=i+1}^{p}(\psi_i-\psi_j)$$

for $\psi_1 \geqq \dots \geqq \psi_p > 0$ *and is* 0 *elsewhere. The distribution of* G *is "uniform."*

PROOF. That the marginal density of $\psi_1, \dots, \psi_p$ is (3.8) has been proved by *Hsu* [5]. It remains to show that G is distributed independently of Ψ and "uniformly." The "uniform" distribution of all orthogonal p-dimensional matrices is given by the (normalized) Haar measure on the orthogonal group; that is, the (normalized) Haar measure is the only probability measure on the group that is invariant under the group operation on the right [4]. Since we require $g_{i1} \geqq 0$, our definition of "uniform distribution" is the conditional distribution obtained from the Haar measure by requiring $g_{i1} \geqq 0$. For this part of the space the probability measure is 2^p times the normalized Haar measure.

The measure on the space of u_f defines a measure on the space of G, $g_{i1} \geqq 0$. Consider any measurable set H in the space of all orthogonal matrices. Let the diagonal matrices with diagonal elements $+1$ and -1 be $J_1, \dots, J_{2^p}$. Let $H = \sum_{i=1}^{2^p} H_i$, where J_iH_i is a set in the space of G, $g_{i1} \geqq 0$. Define the measure of H as the sum of the measures of J_iH_i. Now let us show that this measure is invariant with respect to multiplication on the right. Let E_i be the set in the space of u_f that maps into J_iH_i. Let H^* be HP; that is, H^* is the set obtained by multiplying each element of H on the right by the orthogonal matrix P. Then $H^* = \sum H_i^* = \sum H_iP$. We now show that the measure of H_i^* is the same as H_i. Let $H_i^* = \sum_j H_{ij}^*$ such that $J_jH_{ij}^*$ is in the space $g_{i1} \geqq 0$. Let E_{ij}^* be the set in the space u_f that maps into $J_iH_{ij}^*$. Then $\sum_j E_{ij}^* = P'E_i$; that is, $\sum_j E_{ij}^*$ is the set obtained by multiplying each $(u_1, \dots, u_m)$ by P' on the left. The measure of $P'E_i$ is the integral of the density of $P'u_1, \dots, P'u_m$ over E_i. Since the density of $P'u_1, \dots, P'u_m$ is the same as that of $u_1, \dots, u_m$, the measure of $P'E_i$ is that of E_i. Thus the measure of H^* is that of H. This proves that the measure is invariant with regard to the group operation on the right. Since there is only one such measure on the group of orthogonal matrices with total measure 2^p, this is it. The joint distribution of Φ and $G(g_{i1} \geqq 0)$ has a density. This density does not depend on G because the density at Φ and G is the same as at Φ and G^* since G^* can be obtained from G by multiplication on the right by some orthogonal matrix P and this is equivalent to transforming B to $P'BP$ which has the same characteristic roots as B. This proves the theorem.

Now suppose the density function of $B = B'$ is

$$(3.9)\qquad \pi^{-p(p+1)/4}2^{-p/2}e^{-\mathrm{tr}B^2/2}$$

that is, b_{ij} $(i = 1, \ldots, p; j = i, i + 1, \ldots, p)$ are independently and normally distributed with means zero; the variance of b_{ii} is 1 and that of b_{ij} $(i < j)$ is $\frac{1}{2}$. Now define $\boldsymbol{G}$ and $\boldsymbol{\Psi}$ (diagonal) by (3.3) and (3.4) with the understanding that the elements of the first column of $\boldsymbol{G}$ are nonnegative. The ordered roots ψ_i are not restricted to being nonnegative.

THEOREM 2. *Let the symmetric matrix $\boldsymbol{B}$ have the distribution with density* (3.9). *Then $\boldsymbol{G}$ and $\boldsymbol{\Psi}$, defined by* (3.3), (3.4), *the restriction that $\boldsymbol{\Psi}$ is diagonal with diagonal elements in descending order and $g_{i1} \geqq 0$, are independently distributed. The density of the diagonal elements of $\boldsymbol{\Psi}$ is*

$$2^{-p/2}\left\{\prod_{i=1}^{p}\Gamma\left[\tfrac{1}{2}(p+1-i)\right]\right\}^{-1} e^{-\sum_{i=1}^{p}\psi_i^2/2}\prod_{i=1}^{p}\prod_{j=i+1}^{p}(\psi_i-\psi_j) \tag{3.10}$$

for $\psi_1 \geqq \ldots \geqq \psi_p$ and 0 elsewhere. The distribution of the orthogonal matrix $\boldsymbol{G}$ is uniform.

PROOF. The proof that the marginal density of $\psi_1, \ldots, \psi_p$ is (3.10) has been given by Hsu [8]. The remainder of the proof is the same as for theorem 1 since the density of $\boldsymbol{P'BP}$ for $\boldsymbol{P}$ orthogonal is the same as $\boldsymbol{B}$.

4. An asymptotic distribution when all population roots are zero

A simple case of our main problem is the case where $\tau_g^2(n) = 0$ for all g and n. Then $\boldsymbol{D}_n = \boldsymbol{D}$ has a Wishart distribution with q_2 $(\geqq p)$ degrees of freedom which does not depend on n. In this section we shall find the asymptotic distribution of $\boldsymbol{X}_n$ and $\boldsymbol{\Phi}_n$ in this special case.

In all of the asymptotic theory we use the result [8] that as $n \to \infty$

$$\boldsymbol{U}_n = \frac{1}{\sqrt{n}}(\boldsymbol{A}_n - n\boldsymbol{I}) \tag{4.1}$$

is asymptotically normally distributed with mean zero. The functionally independent variables are statistically asymptotically independent and the variances are given by

$$\mathcal{E}u_{ii}^2 = 2, \quad \mathcal{E}u_{ij}^2 = 1, \qquad i \neq j. \tag{4.2}$$

The matrices $\boldsymbol{X}_n$ and $\boldsymbol{\Phi}_n$ are defined by

$$\frac{1}{n}\boldsymbol{D} = \boldsymbol{X}_n'\boldsymbol{\Phi}_n\boldsymbol{X}_n, \tag{4.3}$$

$$\frac{1}{n}\boldsymbol{A}_n = \boldsymbol{X}_n'\boldsymbol{X}_n, \tag{4.4}$$

where $x_{i1}(n) \geqq 0$, $\boldsymbol{\Phi}_n$ is diagonal and the diagonal elements of $\boldsymbol{\Phi}_n$ are labelled in descending order. For each n, $\boldsymbol{X}_n$ and $\boldsymbol{\Phi}_n$ are defined uniquely except on a set of probability zero.

As $n \to \infty$, $\frac{1}{n}\boldsymbol{A}_n$ approaches the stochastic limit $\boldsymbol{I}$ and $\frac{1}{n}\boldsymbol{D}$ approaches the stochastic limit $\boldsymbol{0}$. In the limit $\boldsymbol{X}_n$ must satisfy

$$\boldsymbol{I} = \boldsymbol{X}'\boldsymbol{X}, \tag{4.5}$$

and $\boldsymbol{\Phi}_n$ must approach $\boldsymbol{0}$ stochastically.

To obtain the full asymptotic theory we define new matrices W_n, Z_n and Θ_n. For any matrix X_n we have an orthogonal matrix O_n and a diagonal matrix Δ_n defined by

$$X_n' X_n = O_n' \Delta_n O_n , \tag{4.6}$$

where the diagonal elements of Δ are ordered in descending size and $o_{i1}(n) \geqq 0$. Let

$$G_n = O_n' \Delta_n^{1/2} O_n , \tag{4.7}$$

where the elements of $\Delta_n^{1/2}$ are the positive square roots of the corresponding elements of Δ_n (the roots are different from 0 when $X_n' X_n$ is nonsingular). Let

$$W_n = X_n G_n^{-1} . \tag{4.8}$$

This is an orthogonal matrix; that is,

$$W_n' W_n = I . \tag{4.9}$$

Let

$$Z_n = \sqrt{n} W_n (G_n - I) . \tag{4.10}$$

Then

$$X_n = W_n G_n = W_n \left(I + W_n' \frac{1}{\sqrt{n}} Z_n \right) = W_n + \frac{1}{\sqrt{n}} Z_n . \tag{4.11}$$

We notice that

$$W_n' Z_n = Z_n' W_n , \tag{4.12}$$

because

$$W_n' Z_n = \sqrt{n} W_n' W_n (G_n - I) = G_n - I = G_n' - I' = \sqrt{n} (G_n' - I) W_n' W_n = Z_n' W_n . \tag{4.13}$$

Now let us show that (4.9), (4.11), and (4.12) define W_n and Z_n in terms of X_n (except for a set of measure 0). We have

$$X_n = W_n O_n' \Delta_n^{1/2} O_n . \tag{4.14}$$

Let W^* be another matrix satisfying (4.9), (4.11) and (4.12), with possibly a different Z_n. Then

$$W^{*\prime} X_n = X_n' W^* , \tag{4.15}$$

$$X_n W^{*\prime} = W^* X_n' . \tag{4.16}$$

Equation (4.15) is

$$W^{*\prime} W_n O_n' \Delta_n^{1/2} O_n = O_n' \Delta_n^{1/2} O_n W_n' W^* . \tag{4.17}$$

From this we derive

$$O_n W^{*\prime} W_n O_n' \Delta_n^{1/2} = \Delta_n^{1/2} O_n W_n' W^* O_n' . \tag{4.18}$$

Let

$$O_n W^{*\prime} W_n O_n' = O^* . \tag{4.19}$$

Then

$$O^* \Delta_n^{1/2} = \Delta_n^{1/2} O^{*\prime} . \tag{4.20}$$

The component equations are

$$o^*_{ij}\delta_j^{1/2} = \delta_i^{1/2}o^*_{ji}\,. \tag{4.21}$$

This gives us

$$o^*_{ij} = \frac{\delta_i^{1/2}}{\delta_j^{1/2}}\, o^*_{ji}\,. \tag{4.22}$$

From (4.16) we derive

$$o^*_{ij} = \frac{\delta_j^{1/2}}{\delta_i^{1/2}}\, o^*_{ji}\,. \tag{4.23}$$

If $\delta_i \neq \delta_j$, $o^*_{ij} = 0$. Therefore, if the δ_i are all different

$$O^* = I\,, \tag{4.24}$$

and

$$W^* = W\,. \tag{4.25}$$

Therefore, except for a set of measure zero of X_n, (4.9), (4.11), and (4.12) define W_n and Z_n uniquely. Let

$$\Theta_n = n\Phi_n\,. \tag{4.26}$$

Now let us substitute into (4.3) and (4.4). We obtain

$$D = W_n'\Theta_n W_n + \frac{1}{\sqrt{n}}(Z_n'\Theta_n W_n + W_n\Theta_n Z_n) + \frac{1}{n} Z_n'\Theta_n Z_n\,, \tag{4.27}$$

$$U_n = W_n'Z_n + Z_n'W_n + \frac{1}{\sqrt{n}} Z_n'Z_n\,. \tag{4.28}$$

Together with (4.9), (4.12) and

$$w_{i1}(n) + \frac{1}{\sqrt{n}} z_{i1}(n) \geqq 0\,, \qquad i = 1, \ldots, p\,, \tag{4.29}$$

(4.27) and (4.28) define Θ_n, W_n and Z_n uniquely for each n.

For given $W_n = W$, $Z_n = Z$ and $\Theta_n = \Theta$ the limits of (4.27) and (4.28) expressing D and U in terms of W_n, Z_n and Θ_n are

$$D = W'\Theta W\,, \tag{4.30}$$

$$U = W'Z + Z'W = 2W'Z\,. \tag{4.31}$$

If

$$w_{i1} \geqq 0\,, \tag{4.32}$$

and $\theta_i > \theta_j$ for $i > j$, then (4.9), (4.12), (4.30) and (4.31) define W, Z, and Θ uniquely in terms of D and U (except for a set of D and U of measure 0). Now we wish to argue that if we take (4.9), (4.12), (4.27), (4.28), and (4.29) as defining W_n, (diagonal) Θ_n, Z_n in terms of (nonrandom) $D = D_n$ and U_n, the limit of W_n, Θ_n and Z_n is the solution of (4.9), (4.12), (4.30), (4.31), and (4.32) as $n \to \infty$ for $D_n \to D$ and $U_n \to U$ where D and U are such that the solution is

unique (the exceptional D and U are of measure 0). A diagonal element of Θ_n is a root of

$$\left| D_n - \theta \left(I + \frac{1}{\sqrt{n}} U_n \right) \right| = 0 . \tag{4.33}$$

As $n \rightarrow \infty$, this root approaches the root of

$$|D - \theta I| = 0 , \tag{4.34}$$

and this is an element of Θ defined by (4.9) and (4.30). Z_n is defined (equivalently) by

$$Z_n = \sqrt{n}\,(X_n - W_n) = X_n O_n' \sqrt{n}\,(I - \Delta_n^{-1/2})\, O_n , \tag{4.35}$$

where the diagonal elements of Δ_n are roots of

$$\left| \frac{1}{\sqrt{n}} U_n + I - \delta I \right| = 0 . \tag{4.36}$$

Let $\psi_i(n)$ be the i-th root of

$$|U_n - \psi I| = 0 . \tag{4.37}$$

Then

$$\delta_i(n) = 1 + \frac{1}{\sqrt{n}} \psi_i(n) . \tag{4.38}$$

Clearly

$$\lim_{n \rightarrow \infty} \sqrt{n}\,[1 - \delta_i^{-1/2}(n)] = \tfrac{1}{2} \lim_{n \rightarrow \infty} \psi_i(n) = \tfrac{1}{2}\psi_i . \tag{4.39}$$

Since $X_n'X_n \rightarrow I$ and O_n is orthogonal, each element of Z_n is bounded in the limit. Thus the norm (any standard norm) of $\frac{1}{\sqrt{n}} Z_n'Z_n$ and the norm of $\frac{1}{\sqrt{n}} (Z_n'\Theta_n W_n + W_n'\Theta_n Z_n) + \frac{1}{n} Z_n'\Theta_n Z_n$ go to zero as $n \rightarrow \infty$. Thus each element of $D_n - W_n'\Theta_n W_n$ and each element of $U_n - 2W_n'Z_n$ goes to 0 as $n \rightarrow \infty$. Consider the matrix function $(P, Q) = (D - W^{*\prime}\Theta^* W^*, U - 2W^{*\prime}Z^*)$, where W^* and Θ^* satisfy our usual conditions including (4.32). The inverse functions W^*, Θ^*, Z^* (as functions of P and Q) are continuous in the proper domain (except on the exceptional set). Hence, if the norm of (P, Q) is sufficiently small the norm of $(W^* - W, \Theta^* - \Theta, Z^* - Z)$ must be arbitrarily small. If $w_{i1} > 0$, then $w_{i1}^* > 0$ for norm of (P, Q) sufficiently small. Then $w_{i1}(n)$ for n sufficiently large is bounded away from 0, and for n sufficiently large $w_{i1}(n)$ satisfying (4.29) must satisfy (4.32). Thus W_n, Θ_n and Z_n converge to W, Θ, and Z defined by (4.30) and (4.31).

The limiting equations (4.30) and (4.31) define W, Z, and Θ uniquely except on a set of Lebesgue measure zero. The discontinuities can only occur on this set. Now considering D and U_n as random matrices we observe that the limiting distribution of D and U_n is absolutely continuous. Thus the conditions of Rubin's theorem are fulfilled. To obtain the limiting distribution of the random matrices W_n, Z_n and Θ_n defined in terms of the random matrices D and U_n we need only find the distribution of W, Z and Θ defined by (4.9), (4.12), (4.30), (4.31) and (4.32), where U has the limiting distribution of U_n.

The distribution of W and Θ is that of theorem 1. The conditional distribution

of Z given W and Θ is obtained from

$$Z = \tfrac{1}{2} W U . \tag{4.40}$$

Thus

$$\mathcal{E}\{Z \mid W\} = \tfrac{1}{2} W \mathcal{E} U = 0 . \tag{4.41}$$

Let

$$\begin{aligned} U &= (u_1, \ldots, u_p) , \\ Z &= (z_1, \ldots, z_p) , \\ W &= (w_1, \ldots, w_p) . \end{aligned} \tag{4.42}$$

Then

$$\mathcal{E}\{z_i z_j' \mid W\} = \tfrac{1}{4} W \mathcal{E}(u_i u_j') W . \tag{4.43}$$

Since $\mathcal{E} u_{ii}^2 = 2$ and $\mathcal{E} u_{ij}^2 = 1$ for $i \neq j$, and $\mathcal{E} u_{ij} u_{kl} = 0$ otherwise, then

$$\begin{aligned} \mathcal{E} u_i u_i' &= I + \varepsilon_{ii} , \\ \mathcal{E} u_i u_j' &= \varepsilon_{ji} \end{aligned} \tag{4.44}$$

where ε_{ij} is a matrix with 1 in the i-th row and j-th column and 0's elsewhere. Thus

$$\begin{aligned} \mathcal{E}\{z_i z_j' \mid W\} &= \tfrac{1}{4} W (I \delta_{ij} + \varepsilon_{ji}) W' \\ &= \tfrac{1}{4} (I \delta_{ij} + w_j w_i') . \end{aligned} \tag{4.45}$$

Since U is normally distributed, the conditional distribution of Z is normal.

THEOREM 3. *Let D have the distribution $W(I, q_2)$, $q_2 \geqq p$, and let A_n be independent of D_n and have the distribution $W(I, n)$. Define X_n and Φ_n by means of* (4.3), (4.4) *and the conditions that $x_{i1}(n) \geqq 0$ and Φ_n is diagonal with diagonal elements in descending order. Let $n\Phi_n = \Theta_n$ and let $X_n = W_n + \frac{1}{\sqrt{n}} Z_n$, where $W_n' W_n = I$ and $W_n' Z_n = Z_n' W_n$. The limiting distribution of Θ_n, W_n and Z_n as $n \to \infty$ is the joint distribution of Θ, $\mathbf{W}$, and $\mathbf{Z}$ such that the marginal distribution of the diagonal matrix Θ and the orthogonal matrix W is that of theorem* 1 *with $m = q_2$ and the conditional distribution of Z given W and Θ is normal with mean* 0 *and covariances given by* (4.45).

5. An asymptotic distribution when all population roots are equal but different from zero

Another special case that is easy to treat is the case of all roots of (2.1) being equal but different from 0, say, $\tau_1^2(n) = \ldots = \tau_p^2(n) = \lambda_n > 0$. Then

$$D_n = F + \sqrt{n} E_n + n \lambda_n I , \tag{5.1}$$

where

$$F = \sum_{g=1}^{q_2} y_g y_g' , \tag{5.2}$$

and E_n is composed of elements

$$\sqrt{\lambda_n} (y_{ij} + y_{ji}) . \tag{5.3}$$

We are interested in X_n and Φ_n (diagonal) defined by

$$\frac{1}{n} F + \frac{1}{\sqrt{n}} E_n + \lambda_n I = X_n' \Phi_n X_n , \tag{5.4}$$

$$\frac{1}{n} A_n = X_n' X_n , \tag{5.5}$$

with $x_{i1}(n) \geq 0$ and the diagonal elements of Φ_n arranged in descending order.

Let

$$X_n = W_n + \frac{1}{\sqrt{n}} Z_n , \tag{5.6}$$

where W_n and Z_n satisfy (4.9) and (4.12). Let

$$\Phi_n = \lambda_n I + \frac{1}{\sqrt{n}} \Theta_n , \tag{5.7}$$

where Θ_n is diagonal. Then (5.4) and (5.5) are

$$\begin{aligned} \frac{1}{n} F + \frac{1}{\sqrt{n}} E_n + \lambda_n I &= \left(W_n + \frac{1}{\sqrt{n}} Z_n \right)' \left(\lambda_n I + \frac{1}{\sqrt{n}} \Theta_n \right) \left(W_n + \frac{1}{\sqrt{n}} Z_n \right) \\ &= \lambda_n I + \frac{1}{\sqrt{n}} [\lambda_n (Z_n' W_n + W_n' Z_n) + W_n' \Theta_n W_n] \\ &\quad + \frac{1}{n} [W_n' \Theta_n Z_n + Z_n' \Theta_n W_n + \lambda_n Z_n' Z_n] + \frac{1}{n^{3/2}} Z_n' \Theta_n Z_n , \end{aligned} \tag{5.8}$$

$$\frac{1}{n} A_n = \left(W_n + \frac{1}{\sqrt{n}} Z_n \right)' \left(W_n + \frac{1}{\sqrt{n}} Z_n \right); \tag{5.9}$$

that is,

$$\frac{1}{\sqrt{n}} (A_n - nI) = (W_n' Z_n + Z_n' W_n) + \frac{1}{\sqrt{n}} Z_n' Z_n . \tag{5.10}$$

Multiply (5.10) by λ_n and subtract from $\sqrt{n}$ times (5.8) to obtain

$$\begin{aligned} \frac{1}{\sqrt{n}} F + E_n - \frac{1}{\sqrt{n}} \lambda_n (A_n - nI) &= W_n' \Theta_n W_n + \frac{1}{\sqrt{n}} [W_n' \Theta_n Z_n + Z_n' \Theta_n W_n] \\ &\quad + \frac{1}{n} Z_n' \Theta_n Z_n . \end{aligned} \tag{5.11}$$

Let

$$U_n = \frac{1}{\sqrt{n}} (A_n - nI) . \tag{5.12}$$

Then (5.10) and (5.11) can be written as

$$U_n = (W_n' Z_n + Z_n' W_n) + \frac{1}{\sqrt{n}} Z_n' Z_n , \tag{5.13}$$

$$\frac{1}{\sqrt{n}} F + E_n - \lambda_n U_n = W_n' \Theta_n W_n + \frac{1}{\sqrt{n}} [W_n' \Theta_n Z_n + Z_n' \Theta_n W_n] + \frac{1}{n} Z_n' \Theta_n Z_n , \tag{5.14}$$

where

$$W_n' W_n = I , \tag{5.15}$$

$$W_n' Z_n = Z_n' W_n , \tag{5.16}$$

$$(5.17)\qquad w_{i1}(n) + \frac{1}{\sqrt{n}} z_{i1}(n) \geqq 0\,, \qquad i = 1, \ldots, p\,.$$

Thus for a given n, Θ_n, W_n and Z_n are defined as functions of U_n, $F = F_n$ and E_n. The functions are unique and continuous except over a set of U_n, F_n and E_n of measure zero. The limit of the functions (as $U_n \to U$, $F_n \to F$, and $E_n \to E$) is the solution to

$$(5.18)\qquad U = W'Z + Z'W = 2W'Z\,,$$

$$(5.19)\qquad E - \lambda U = W'\Theta W\,,$$

with W satisfying (4.32) and where $\lambda = \lim_{n\to\infty} \lambda_n$. This argument is justified as in section 3. In particular, each diagonal element of Θ_n as a function of nonrandom F_n, E_n and U_n is a root of

$$(5.20)\qquad \left| \frac{1}{\sqrt{n}} F_n + E_n + \sqrt{n}\lambda_n I - (U_n + \sqrt{n} I)\left(\lambda_n + \frac{1}{\sqrt{n}}\,\theta\right) \right| = 0\,,$$

that is, of

$$(5.21)\qquad \left| \left(\frac{1}{\sqrt{n}} F_n + E_n - \lambda_n U_n\right)\left(I + \frac{1}{\sqrt{n}} U_n\right)^{-1} - \theta I \right| = 0\,.$$

Since

$$(5.22)\qquad \left(\frac{1}{\sqrt{n}} F_n + E_n - \lambda_n U_n\right)\left(I + \frac{1}{\sqrt{n}} U_n\right)^{-1} \to E - \lambda U\,,$$

the ordered roots of (5.20) approach the ordered roots of

$$(5.23)\qquad |E - \lambda U - \theta I| = 0\,.$$

As in section 4 we can argue that the elements of W_n, Θ_n and Z_n are bounded for $F_n \to F$, $E_n \to E$ and $U_n \to U$. Then the elements in (5.13) and (5.14) which are multiplied by $1/\sqrt{n}$, $1/n$ and $1/n^{3/2}$ go to zero. The remainder of the argument of section 4 applies.

We can now use Rubin's theorem since the discontinuities of the mapping occur where there is indeterminacy and the set of E and U where this occurs is of limiting probability 0. To find the limiting distribution of the random matrices W_n, Θ_n, and Z_n we need only consider the distribution of the random matrices W, Θ, and Z defined by (5.18), (5.19) and (4.32) (with W orthogonal, Θ diagonal with diagonal elements in descending order), where the random matrices $\mathbf{E}$ and U have the limiting distribution of E_n and U_n. Let

$$(5.24)\qquad E - \lambda U = V\,.$$

The density of E and U is

$$\begin{aligned}(5.25)\qquad k\,e^{-\{\operatorname{tr} E^2/(4\lambda) + \operatorname{tr} U^2/2\}/2} &= k\,e^{-\operatorname{tr}\{(V+\lambda U)^2 + 2\lambda U^2\}/(8\lambda)} \\ &= k\,e^{-\operatorname{tr}\{V^2 + 2\lambda UV + \lambda^2 U^2 + 2\lambda U^2\}/(8\lambda)} \\ &= k\,e^{-\operatorname{tr}\{(\lambda^2+2\lambda)(U + [\lambda/(\lambda^2+2\lambda)]V)^2 + [2\lambda/(\lambda^2+2\lambda)]V^2\}/(8\lambda)} \\ &= k\,e^{-(\lambda+2)\operatorname{tr}(U + [1/(\lambda+2)]V)^2/8}\,e^{-\operatorname{tr} V^2/(4\lambda^2+8\lambda)}.\end{aligned}$$

This marginal density of V is normal, and the conditional density of U is normal. The distribution of W and $(2\lambda^2 + 4\lambda)^{-1/2}\Theta$ is that of theorem 2. The conditional

distribution of Z given W and Θ is normal with mean

$$\mathcal{E}\{Z|W,\Theta\}=\tfrac{1}{2}W\mathcal{E}\{U|W,\Theta\}=\tfrac{1}{2}W\left(-\frac{1}{\lambda+2}V\right)=-\frac{1}{2(\lambda+2)}\Theta W. \tag{5.26}$$

The conditional covariance between two columns of Z is

$$\begin{aligned}\mathcal{E}\{[z_i-\mathcal{E}(z_i|W,\Theta)][z_j-\mathcal{E}(z_j|W,\Theta)]'|W,\Theta\}\\ =\tfrac{1}{4}W\mathcal{E}\left\{\left(u_i+\frac{1}{2+\lambda}v_i\right)\left(u_j+\frac{1}{2+\lambda}v_j\right)'\middle|V\right\}W'\\ =\frac{1}{2\lambda+4}(I\delta_{ij}+w_jw_i').\end{aligned} \tag{5.27}$$

THEOREM 4. *Let*

$$D_n=\sum_{g=1}^{q_2}(y_g+\sqrt{n}\sqrt{\lambda}\,\varepsilon_g)(y_g+\sqrt{n}\sqrt{\lambda_n}\varepsilon_g)' \tag{5.28}$$

where the p-dimensional vectors $y_1,\ldots,y_{q_2}$ $(q_2\geqq p)$ *are independently distributed according to* $N(0,I)$; *let* A_n *be independently distributed according to* $W(I,n)$. *Define* X_n *and* Φ_n *by means of* (4.3), (4.4), *and the conditions that* $x_{i1}(n)\geqq 0$ *and* Φ_n *is diagonal with diagonal elements in descending order. Let* $\Theta_n=\sqrt{n}(\Phi_n-\lambda_n I)$ *and* $X_n=W_n+\frac{1}{\sqrt{n}}Z_n$, *where* $W_n'W_n=I$, *and* $W_n'Z_n=Z_n'W_n$. *The limiting distribution of* Θ_n, W_n *and* Z_n *as* $n\to\infty$ *is the joint distribution of* Θ, W *and* Z *such that the marginal distribution of* $(2\lambda^2+4\lambda)^{-1/2}$ *times the diagonal elements of the diagonal matrix* Θ *and the orthogonal matrix* W *is that of theorem* 2 *and the conditional distribution of* Z *given* W *and* Θ *is normal with mean* (5.26) *and covariances* (5.27).

6. An asymptotic distribution in the general case

Now we consider the general case of the population roots having different values. We assume that the multiplicities do not depend on n, but the values may. Let

$$[\tau_i^2(n)\,\delta_{ij}]=\begin{pmatrix}\lambda_1(n)I & 0 & \cdots & 0 & 0\\ 0 & \lambda_2(n)I & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda_h(n)I & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}=\Lambda_n\,, \tag{6.1}$$

say, where $\lambda_i(n)I$ is of order r_i and $r_{h+1}=p-\sum_{i=1}^{h}r_i$ is the multiplicity of 0. Partition X_n, F, E_n and U_n similarly, to $[\tau_i^2(n)\delta_{ij}]$. Let

$$X_n=\begin{pmatrix}(W_1n) & 0 & \cdots & 0\\ 0 & W_2(n) & \cdots & 0\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ 0 & 0 & \cdots & W_{h+1}(n)\end{pmatrix}+\frac{1}{\sqrt{n}}\begin{pmatrix}Z_{11}(n) & Z_{12}(n) & \cdots & Z_{1,h+1}\\ Z_{21}(n) & Z_{22}(n) & \cdots & Z_{2,h+1}\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ Z_{h+1,1} & Z_{h+1,2} & \cdots & Z_{h+1,h+1}\end{pmatrix} \tag{6.2}$$

$$=W_n+\frac{1}{\sqrt{n}}Z_n$$

where

$$W_i(n)Z'_{ii}(n) = Z_{ii}(n)W'_i(n). \tag{6.3}$$

As before, this defines $W_i(n)$ and $Z_{ii}(n)$ uniquely in terms of $X_{ii}(n)$. To make X_n unique in this case we now require that the elements of the first column of $X_{ii}(n)$ be nonnegative.

Let

(6.4)

$$\Phi_n = \begin{pmatrix} \lambda_1(n)I + \frac{1}{\sqrt{n}}\Theta_1(n) & 0 & \dots & 0 & 0 \\ 0 & \lambda_2(n)I + \frac{1}{\sqrt{n}}\Theta_2(n) & \dots & 0 & 0 \\ \\ 0 & 0 & \dots & \lambda_h(n)I + \frac{1}{\sqrt{n}}\Theta_h(n) & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{n}\Theta_{h+1}(n) \end{pmatrix}.$$

Then

$$\frac{1}{n}D_n = \frac{1}{n}F + \frac{1}{\sqrt{n}}E_n + \Lambda = X'_n\Phi_n X_n, \tag{6.5}$$

$$\frac{1}{n}A_n = \frac{1}{\sqrt{n}}U_n + I = X'_n X_n. \tag{6.6}$$

The submatric equations of (6.6) are

$$\begin{aligned} \frac{1}{\sqrt{n}}U_{ii}(n) + I &= \sum_j X'_{ji}(n)X_{ji}(n) \\ &= W'_i(n)W_i(n) + \frac{1}{\sqrt{n}}[W'_i(n)Z_{ii}(n) + Z'_{ii}(n)W_i(n)] \\ &\qquad + \frac{1}{n}\sum_j Z'_{ji}(n)Z_{ji}(n). \end{aligned} \tag{6.7}$$

$$\begin{aligned} \frac{1}{\sqrt{n}}U_{ij}(n) &= \sum_k X'_{ki}(n)X_{kj}(n) \\ &= \frac{1}{\sqrt{n}}[W'_i(n)Z_{ij}(n) + Z'_{ji}(n)W_j(n)] \\ &\qquad + \frac{1}{n}\sum_k Z'_{ki}(n)Z_{kj}(n), \qquad i \neq j. \end{aligned} \tag{6.8}$$

The submatric equations of (6.5) are

$$\begin{aligned} \frac{1}{n}F_{ii} + \frac{1}{\sqrt{n}}E_{ii}(n) + \lambda_i(n)I &= \sum_k X'_{ki}(n)\Phi_k(n)X_{ki}(x) \\ &= \sum_{k=1}^{h}\left(\delta_{ki}W'_i(n) + \frac{1}{\sqrt{n}}Z'_{ki}(n)\right)\left(\lambda_k(n)I + \frac{1}{\sqrt{n}}\Theta_k(n)\right) \\ &\quad \times\left(\delta_{ki}W_i(n) + \frac{1}{\sqrt{n}}Z_{ki}(n)\right) + \frac{1}{n^2}Z'_{h+1,i}(n)\Theta_{h+1}(n)Z_{h+1,i}(n) \\ &= \lambda_i(n)W'_i(n)W_i(n) + \frac{1}{\sqrt{n}}\{W'_i(n)\Theta_iW_i(n) + \lambda_i(n) \times \end{aligned} \tag{6.9}$$

$$\times [W_i'(n) Z_{ii}(n) + Z_{ii}'(n) W_i(n)]\} + \frac{1}{n}\Big[\sum_{k=1}^{h} \lambda_k(n) Z_{ki}'(n) Z_{ki}(n)$$

$$+ W_i'(n)\, \Theta_i(n)\, Z_{ii}(n) + Z_{ii}'(n)\, \Theta_i(n)\, W_i(n)\Big]$$

$$+ \frac{1}{n^{3/2}} \sum_{k=1}^{h} Z_{ki}'(n)\, \Theta_k(n)\, Z_{ki}(n) + \frac{1}{n^2} Z_{h+1,i}'(n)\, \Theta_{h+1}(n) Z_{h+1,i}(n), \qquad i \neq h+1\,,$$

$$(6.10)\quad \frac{1}{n} F_{h+1,h+1} = \sum_k X_{k,h+1}'(n)\, \Phi_k(n) X_{k,h+1}(n)$$

$$= \frac{1}{n}\Big[\sum_{k=1}^{h} Z_{k,h+1}'(n)\Big(\lambda_k(n) I + \frac{1}{\sqrt{n}}\, \Theta_k(n)\Big) Z_{k,h+1}'(n) + \Big(W_{h+1}'(n)$$

$$+ \frac{1}{\sqrt{n}} Z_{h+1,h+1}'(n)\Big)\Theta_{h+1}(n)\Big(W_{h+1}(n) + \frac{1}{\sqrt{n}} Z_{h+1,h+1}(n)\Big)\Big]$$

$$= \frac{1}{n}\Big[W_{h+1}'(n)\, \Theta_{h+1}(n)\, W_{h+1}(n) + \sum_{k=1}^{h} \lambda_k(n)\, Z_{k,h+1}'(n)\, Z_{k,h+1}(n)\Big]$$

$$+ \frac{1}{n^{3/2}}\Big[\sum_{k=1}^{h} Z_{k,h+1}'(n)\, \Theta_k(n) Z_{k,h+1}(n) + W_{h+1}'(n)\, \Theta_{h+1}(n) Z_{h+1,h+1}(n)$$

$$+ Z_{h+1,h+1}'(n)\, \Theta_{h+1}(n) W_{h+1}(n)\Big] + \frac{1}{n^2} Z_{h+1,h+1}'(n)\, \Theta_{h+1}(n) Z_{h+1,h+1}(n),$$

$$(6.11)\quad \frac{1}{n} F_{ij} + \frac{1}{\sqrt{n}} E_{ij}(n) = \sum_k X_{ki}'(n)\, \Phi_k(n) X_{kj}(n)$$

$$= \sum_{k=1}^{h}\Big(\delta_{ki} W_i'(n) + \frac{1}{\sqrt{n}} Z_{ki}'(n)\Big)\Big(\lambda_k(n) I + \frac{1}{\sqrt{n}}\, \Theta_k(n)\Big)$$

$$\times\Big(\delta_{kj} W_j(n) + \frac{1}{\sqrt{n}} Z_{kj}(n)\Big) + \frac{1}{n^2} Z_{h+1,i}'(n)\, \Theta_{h+1}(n)\, Z_{h+1,j}(n)$$

$$= \frac{1}{\sqrt{n}}[\lambda_i(n)\, W_i'(n) Z_{ij}(n) + \lambda_j(n) Z_{ji}'(n)\, W_j(n)]$$

$$+ \frac{1}{n}\Big[\sum_{k=1}^{h} \lambda_k(n) Z_{ki}'(n) Z_{kj}(n) + W_i'(n)\, \Theta_i(n) Z_{ij}(n) + Z_{ji}'(n)\, \Theta_j(n)$$

$$\times W_j(n)\Big] + \frac{1}{n^{3/2}}\Big[\sum_{k=1}^{h} Z_{ki}'(n)\, \Theta_k(n) Z_{kj}(n)\Big]$$

$$+ \frac{1}{n^2} Z_{h+1,i}'(n)\, \Theta_{h+1}(n) Z_{h+1,j}(n), \qquad i \neq i\,;\; i,\, j \neq h+1\,,$$

$$(6.12)\quad \frac{1}{n} F_{i,h+1} + \frac{1}{\sqrt{n}} E_{i,h+1}(n) = \sum_{k=1}^{h+1} X_{ki}'(n)\, \Phi_k(n) X_{k,h+1}(n)$$

$$= \sum_{k=1}^{h}\Big(\delta_{ki} W_i'(n) + \frac{1}{\sqrt{n}} Z_{ki}'(n)\Big)\Big(\lambda_k(n) I + \frac{1}{\sqrt{n}}\, \Theta_k(n)\Big)\frac{1}{\sqrt{n}} Z_{k,h+1}(n) +$$

$$+\frac{1}{\sqrt{n}} Z'_{h+1,i}(n)\frac{1}{n}\,\Theta_{h+1}(n)\Big(W_{h+1}(n)+\frac{1}{\sqrt{n}}Z_{h+1,h+1}(n)\Big)$$

$$=\frac{1}{\sqrt{n}}\lambda_i(n)W'_i(n)Z_{i,h+1}(n)+\frac{1}{n}\Big[W'_i(n)\,\Theta_i(n)Z_{i,h+1}(n)+\sum_{k=1}^{h}\lambda_k(n)$$

$$\times Z'_{ki}(n)Z_{k,h+1}(n)\Big]+\frac{1}{n^{3/2}}\Big[\sum_{k=1}^{h}Z'_{ki}(n)\,\Theta_k(n)Z_{k,h+1}(n)+Z'_{h+1,i}(n)$$

$$\times\,\Theta_{h+1}(n)W_{h+1}(n)\Big]+\frac{1}{n^2}Z'_{h+1,i}(n)\,\Theta_{h+1}(n)Z_{h+1,h+1}(n),\qquad i\neq h+1\,.$$

For fixed F, $E_n = E$ and $U_n = U$ (in the proper domain) the above equations define the orthogonal $W_i(n)$, $Z_i(n)$ and the diagonal $\Theta_i(n)$ uniquely (except for a set F, E and U of measure zero) under the restrictions that the elements in the first column of $W_i(n)+\frac{1}{\sqrt{n}}Z_{ii}(n)$ are nonnegative and that the diagonal elements of the $\Theta_i(n)$ are in descending order. Now subtract I from each side of (6.7) and $\lambda_i(n)I$ from each side of (6.9) and multiply (6.7), (6.8), (6.9), (6.11) and (6.12) by $\sqrt{n}$ and (6.10) by n and let $n\to\infty$. Using the fact that $W_i(n)$ is orthogonal and (6.3), we obtain the limiting equations [for $\lambda_i(n)\to\lambda_i$, $E_n\to E$, and $U_n\to U$]

$$U_{ii} = 2W'_iZ_{ii}\,, \tag{6.13}$$

$$U_{ij} = W'_iZ_{ij} + Z'_{ji}W_j\,,\qquad i\neq j\,, \tag{6.14}$$

$$E_{ii} = W'_i\Theta_iW_i + 2\lambda_iW'_iZ_{ii}\,,\qquad i\neq h+1\,, \tag{6.15}$$

$$F_{h+1,h+1} = W'_{h+1}\Theta_{h+1}W_{h+1} + \sum_{k=1}^{h}\lambda_kZ'_{k,h+1}Z_{k,h+1}\,, \tag{6.16}$$

$$E_{ij} = \lambda_iW'_iZ_{ij} + \lambda_jZ'_{ji}W_j\,,\qquad i\neq j;\ i,j\neq h+1\,, \tag{6.17}$$

$$E_{i,h+1} = \lambda_iW'_iZ_{i,h+1}\,,\qquad i\neq h+1\,. \tag{6.18}$$

From (6.13) and (6.15) we obtain

$$E_{ii} - \lambda_iU_{ii} = W'_i\Theta_iW_i\,. \tag{6.19}$$

From (6.16) and (6.18) we obtain

$$F_{h+1,h+1} - \sum_{i=1}^{h}\frac{1}{\lambda_i}E'_{i,h+1}E_{i,h+1} = W'_{h+1}\Theta_{h+1}W_{h+1}\,. \tag{6.20}$$

Then the fact that W_i is orthogonal and the requirement that the elements of the first column of W_i be nonnegative define W_i, Z_{ij}, and Θ_i uniquely.

To show that W_i, Z_{ij}, and Θ_i defined by (6.13), (6.14), (6.17), (6.18), (6.19) and (6.20) are the limits of the matrices defined by (6.7)–(6.12) is more complicated than the similar demonstration in section 4. We shall only sketch this proof briefly. First we should like to prove that the diagonal elements of $\Theta_1(n)$, for

example, converge to the characteristic roots of $E_{11} - \lambda_1 U_{11}$ as $F_n \to F$, $E_n \to E$ and $U_n \to U$. From the equation

$$\left| \frac{1}{n} F_n + \frac{1}{\sqrt{n}} E_n + \Lambda_n - \phi \left(\frac{1}{\sqrt{n}} U_n + I \right) \right| = 0 \tag{6.21}$$

we can show that the first r_1 elements of Φ_n converge to λ_1. Then we need to show that the largest root of (6.21) minus $\lambda_1(n)$ times $\sqrt{n}$ converges to the largest characteristic root of $E_{11} - \lambda_1 U_{11}$. That can be argued from the determinantal equation

$$\left| \frac{1}{n} F_n + \frac{1}{\sqrt{n}} E_n + \Lambda_n - \left(\frac{1}{\sqrt{n}} U_n + I \right) \left(\lambda_1(n) I + \frac{1}{\sqrt{n}} \theta I \right) \right| \tag{6.22}$$

$$= \left| \frac{1}{n} F_n + \frac{1}{\sqrt{n}} [E_n - \lambda_1(n) U_n] + (\Lambda_n - \lambda_1(n) I) - \frac{1}{\sqrt{n}} \theta \left(\frac{1}{\sqrt{n}} U_n + I \right) \right| = 0 .$$

In the second determinant above we can factor out $\sqrt{n}$ from the first r_1 rows. Then as $n \to \infty$ there are r_1 sequences of roots each of which converges to a characteristic root of $E_{11} - \lambda_1 U_{11}$. Similar arguments suffice for $\Theta_1(n)$ $(i \neq h+1)$. For $\Theta_{h+1}(n)$ we can use a slightly more complicated demonstration.[3]

Next we wish to argue that the elements of $Z_{ij}(n)$ are bounded as $n \to \infty$ (as $F_n \to F$, $E_n \to E$, and $U_n \to U$). For convenience let us take the case of $r_i = 1$. Then the characteristic vector say $c_1(n)$ associated with the largest root θ_1 satisfies

$$\left[\frac{1}{n} F_n + \frac{1}{\sqrt{n}} [E_n - \lambda_1(n) U_n] + [\Lambda_n - \lambda_1(n) I] - \frac{1}{\sqrt{n}} \theta_1 \left(\frac{1}{\sqrt{n}} U_n + I \right) \right] c = 0 . \tag{6.23}$$

Then the components of $c_1(n)$ are

$$c_{11}(n) = k(n) \left[\prod_{i=2}^{p} [r_i^2(n) - \lambda_1(n)] + \frac{1}{\sqrt{n}} k_1(n) \right], \tag{6.24}$$

$$c_{1i}(n) = k(n) \frac{1}{\sqrt{n}} k_i(n), \qquad i \neq 1, \tag{6.25}$$

where $k(n)$ approaches a finite limit and $k_i(n)$ are bounded. Using the same reasoning for each characteristic vector (assuming $r_i = 1$) we show that C_n is a diagonal matrix with bounded elements plus $1/\sqrt{n}$ times a matrix with bounded elements. Thus $X_n = C_n^{-1}$ is of the same form. Therefore, $\sqrt{n} x_{ij}(n) = z_{ij}(n)$ $(i \neq j)$ are bounded. From

$$\frac{1}{\sqrt{n}} u_{ii}(n) + 1 = \sum_{j=1}^{p} x_{ij}^2(n) \tag{6.26}$$

$$= x_{ii}^2(n) + \sum_{j \neq i} x_{ij}^2(n)$$

we see that $\sqrt{n}[1 - x_{ii}^2(n)] = - u_{ii}(n) + \sqrt{n} x_{ij}^2(n)$ is bounded. Thus $z_{ii}(n) = \sqrt{n}[1 - x_{ii}(n)]$ is also bounded. If $r_1 \neq 1$, essentially the same argument can be carried out in terms of the partitioned matrices. Thus the norms of matrices such as $U_{ii}(n) - 2W_1'(n) Z_{ii}(n)$ go to zero. The argument of section 4 shows that $Z_{ii}(n)$ approaches Z_{ii}, etc.

[3] Since these arguments are similar to Hsu's [8], it is unnecessary to go into more detail.

We are now in a position to apply Rubin's theorem. The discontinuities occur where $\boldsymbol{W}$, $\boldsymbol{Z}$, and $\boldsymbol{\Theta}$ are not defined uniquely and the measure of $\boldsymbol{U}$, $\boldsymbol{F}$ and $\boldsymbol{E}$ where this occurs is zero. Let the limiting distribution of the random matrices $\boldsymbol{U}_n$, $\boldsymbol{F}$, and $\boldsymbol{E}_n$ be the distribution of the random matrices $\boldsymbol{U}$, $\boldsymbol{F}$, and $\boldsymbol{E}$. Then the limiting distribution of the random matrices $\boldsymbol{W}_n$ (orthogonal diagonal blocks), $\boldsymbol{Z}_n$ and $\boldsymbol{\Theta}_n$ (diagonal) defined by (6.7)–(6.12) is the distribution of $\boldsymbol{W}$ (orthogonal diagonal blocks) $\boldsymbol{Z}$, and $\boldsymbol{\Theta}$ (diagonal) defined in terms of the random matrices $\boldsymbol{U}$, $\boldsymbol{F}$, and $\boldsymbol{E}$ by (6.13), (6.14), (6.17)–(6.20). The distribution of $\boldsymbol{E}_{ii} - \lambda_i \boldsymbol{U}_{ii}$, $i \neq h+1$, is that of section 5. Hence, the distribution of $\boldsymbol{W}_i$ and $\boldsymbol{\Theta}_i$ is that given there. Since $\boldsymbol{E}_{ii} - \lambda_i \boldsymbol{U}_{ii}$ is independent of $\boldsymbol{E}_{jj} - \lambda_j \boldsymbol{U}_{jj}$, $i \neq j$, the matrices $\boldsymbol{W}_i$ and $\boldsymbol{\Theta}_i$, $i = 1, \ldots, h$ are independent. The conditional distribution of $\boldsymbol{Z}_{ii}$ given $\boldsymbol{W}_i$ is also that of section 5 with $\lambda = \lambda_i$, $\boldsymbol{\Theta} = \boldsymbol{\Theta}_i$, $\boldsymbol{W} = \boldsymbol{W}_i$ and $p = r_i$.

Now, consider (6.20). An element of $\boldsymbol{E}_{i,h+1}$ is $e_{gk} = \sqrt{\lambda_i}\, y_{kg}$ $(r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i;\ \ p - r_{h+1} + 1 \leqq k \leqq p)$; an element of $\boldsymbol{F}_{h+1,h+1}$ is $f_{kk'} = \sum_{g=1}^{q_2} y_{kg} y_{k'g}$. Thus an element of $\boldsymbol{F}_{h+1,h+1} - \sum_{i=1}^{h} \frac{1}{\lambda_i} \boldsymbol{E}'_{i,h+1} \boldsymbol{E}_{i,h+1}$ is $\sum_{g=1}^{q_2} y_{kg} y_{k'g} - \sum_{g=1}^{p-r_{h+1}} y_{kg} y_{k'g} = \sum_{g=p-r_{h+1}+1}^{q_2} y_{kg} y_{k'g}$. This matrix of order r_{h+1} has the distribution $W(\boldsymbol{I}, q_2 - p + r_{h+1})$ and is independent of $\boldsymbol{E}_{ii}$, $i \neq h+1$, and $\boldsymbol{E}_{ij}$, $i \neq j$. The distribution of $\boldsymbol{W}_{h+1}$, $\boldsymbol{\Theta}_{h+1}$ and $\boldsymbol{Z}_{h+1,h+1}$ is that of section 4 with $\boldsymbol{W} = \boldsymbol{W}_{h+1}$, $\boldsymbol{\Theta} = \boldsymbol{\Theta}_{h+1}$, $\boldsymbol{Z} = \boldsymbol{Z}_{h+1,h+1}$, $p = r_{h+1}$ and $m = q_2 - p + r_{h+1}$.

The matrices $\boldsymbol{U}_{ij}$ and $\boldsymbol{E}_{ij}$, $i \neq j$, are independent of the other submatrices (except $\boldsymbol{E}_{ji} = \boldsymbol{E}'_{ij}$ and $\boldsymbol{U}_{ji} = \boldsymbol{U}'_{ij}$). From (6.14) and (6.17) or (6.18), we obtain

$$\boldsymbol{E}_{ij} - \lambda_j \boldsymbol{U}_{ij} = (\lambda_i - \lambda_j) \boldsymbol{W}'_i \boldsymbol{Z}_{ij} . \tag{6.27}$$

The conditional distribution of $\boldsymbol{Z}_{ij}$ given $\boldsymbol{W}_i$ is that of

$$\frac{1}{\lambda_i - \lambda_j} \boldsymbol{W}_i (\boldsymbol{E}_{ij} - \lambda_j \boldsymbol{U}_{ij}) . \tag{6.28}$$

An element of $\boldsymbol{E}_{ij}$ is

$$e_{gf} = \sqrt{\lambda_j}\, y_{gf} + \sqrt{\lambda_i}\, y_{fg}\,, \quad r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i\,, \quad r_1 + \ldots + r_{j-1} + 1 \leqq f \leqq r_1 + \ldots + r_j\,. \tag{6.29}$$

Thus an element of $\boldsymbol{E}_{ij} - \lambda_j \boldsymbol{U}_{ij}$ is normally distributed independently of the other elements with zero mean and variance $\lambda_i + \lambda_j + \lambda_j^2$. The conditional distribution of $\boldsymbol{Z}_{ij}$ given $\boldsymbol{W}_i$ is normal with mean $\boldsymbol{0}$ and covariance between two elements is

$$\begin{aligned} \mathcal{E}\{ z_{gf} z_{g'f'} \mid W_i \} &= \frac{1}{(\lambda_i - \lambda_j)^2} \mathcal{E} \sum_{k,k'} w_{gk}^{(i)} (e_{kf} - \lambda_j u_{kf}) w_{g'k}^{(i)} (e_{k'f'} - \lambda_j u_{k'f'}) \\ &= \frac{1}{(\lambda_i - \lambda_j)^2} \sum_{k,k'} w_{gk}^{(i)} w_{g'k'}^{(i)} \delta_{ff'} \delta_{kk'} (\lambda_i + \lambda_j + \lambda_j^2) \\ &= \frac{\delta_{ff'} \delta_{gg'}}{(\lambda_i - \lambda_j)^2} (\lambda_i + \lambda_j + \lambda_j^2) . \end{aligned} \tag{6.30}$$

Thus the variance of z_{gf} is $(\lambda_i + \lambda_j + \lambda_j^2)/(\lambda_i - \lambda_j)^2$ and the covariances are zero.

The conditional distribution of Z_{ji} is similar except that i and j are interchanged. Z_{ij} and Z_{ji} are independent of the other matrices. Now consider the conditional covariance between an element z_{gf} of Z_{ij} and $z_{g'f'}$ of Z_{ji}. It is

$$
\begin{aligned}
(6.31)\quad \mathcal{E}\{ z_{gf} z_{g'f'} \mid W_i, W_j\} &= \frac{-1}{(\lambda_i - \lambda_j)^2} \mathcal{E} \sum_{k,k'} w_{gk}^{(i)} (e_{kf} - \lambda_j u_{kf}) \\
&\qquad \times w_{g'k'}^{(i)} (e_{f'k'} - \lambda_i u_{f'k'}) \\
&= -\frac{\lambda_i + \lambda_j + \lambda_i\lambda_j}{(\lambda_i - \lambda_j)^2} \sum_{k,k'} w_{gk}^{(i)} w_{g'k'}^{(j)} \delta_{kf'} \delta_{fk'} \\
&= -\frac{\lambda_i + \lambda_j + \lambda_i\lambda_j}{(\lambda_i - \lambda_j)^2} w_{gf'}^{(i)} w_{g'f}^{(j)} .
\end{aligned}
$$

We can indicate the conditional covariances between the columns $Z = (z_1, \ldots, z_p)$ in matrix form. If $g, f \leqq r_1$, the conditional covariance between z_g and z_f is

$$
(6.32)\quad \mathcal{E}\{ z_g z_f' \mid \Theta, W\} =
$$

$$
\begin{pmatrix}
\frac{1}{2\lambda_1 + 4} (I\delta_{gf} + w_f^{(1)} w_g^{(1)\prime}) & 0 & \cdots & 0 \\
0 & \frac{\lambda_1 + \lambda_2 + \lambda_2^2}{(\lambda_1 - \lambda_2)^2} I\delta_{gf} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & & \frac{\lambda_1}{\lambda_1^2} I\delta_{gf}
\end{pmatrix} .
$$

If $g \leqq r_1$ and $r_1 + 1 \leqq f \leqq r_1 + r_2$, the conditional covariance is

$$
(6.33)\quad \mathcal{E}\{ z_g z_f' \mid \Theta, W\} =
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
\frac{\lambda_1 + \lambda_2 + \lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^2} w_f^{(2)} w_g^{(1)\prime} & 0 & \cdots & 0 \\
\vdots & \vdots & & \\
0 & 0 & \cdots & 0
\end{pmatrix} .
$$

THEOREM 5. *Let*

$$
(6.34)\quad D_n = \sum_{g=1}^{q_2} [y_g + \sqrt{n}\, \tau_g(n) \varepsilon_g][y_g + \sqrt{n}\, \tau_g(n) \varepsilon_g]' ,
$$

where the p-dimensional vectors $y_1, \ldots, y_{q_2}$ $(q_2 \geqq p)$ *are independently distributed according to* $N(0, I)$ *and* $\tau_g(n) = \sqrt{\lambda_i(n)} \rightarrow \sqrt{\lambda_i}$, $r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i$; $\lambda_i > \lambda_j$, $i < j$ $(i, j = 1, \ldots, h+1)$; $\lambda_{h+1}(n) = 0$. *Let* A_n *be independently distributed according to* $W(I, n)$. *Define* X_n *and* Φ_n *by means of* (4.3), (4.4) *and the conditions that* $x_{g, r_1 + \ldots + r_{i-1} + 1}(n) \geqq 0$, $r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i$, *and* Φ_n *is diagonal with diagonal elements in descending order. Let each of* X_n *and* Φ_n *be partitioned into* $(h+1)^2$ *submatrices of* $r_1, \ldots, r_{h+1}$ *rows and*

$r_1, \ldots, r_{h+1}$ *columns. Let* Θ_n *be defined by* (6.4) *and* W_n *and* Z_n *defined by* (6.2), (6.3) *and* $W_i(n)'W_i(n) = I$. *The limiting distribution of* Θ_n, W_n *and* Z_n *is that of* Θ, W *and* Z *similarly partitioned, which can be described as follows: The matrices* Θ_i, W_i, Z_{ii} *are independent of* Θ_j, W_j, Z_{jj} $(i \neq j)$; *the distribution of* Θ_i, W_i *and* Z_{ii}, $i \neq h+1$, *is that of* Θ, W *and* Z *given in theorem* 4 *with* $p = r_i$; *the distribution of* Θ_{h+1}, W_{h+1} *and* $Z_{h+1,h+1}$ *is that of* Θ, W *and* Z *given in theorem* 3 *with* $p = r_{h+1}$, $m = q_2 - p + r_{h+1}$; *the conditional distribution of* Z *given* Θ *and* W *is normal; the conditional expectation of* Z_{ij} $(i \neq j)$ *is* 0; *the conditional covariance matrix of* z_g *and* z_f', $r_1 + \ldots + r_{i-1} + 1 \leqq g$, $f \leqq r_1 + \ldots + r_i$, *consists of nondiagonal blocks of zeros and the i-th diagonal block is*

$$\frac{1}{2\lambda_i + 4}(I\delta_{gf} + w_f^{(i)} w_g^{(i)\prime})$$

and the j-th diagonal block $(j \neq i)$ *is*

$$\frac{\lambda_i + \lambda_j + \lambda_j^2}{(\lambda_i - \lambda_j)^2} I\delta_{gf};$$

the covariance matrix of z_g *and* z_f', $r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i$, $r_1 + \ldots + r_{j-1} + 1 \leqq f \leqq r_1 + \ldots + r_j$ $(i \neq j)$ *consists of* 0's *except the j-th i-th block which is*

$$\frac{\lambda_i + \lambda_j + \lambda_i\lambda_j}{(\lambda_i - \lambda_j)^2} w_f^{(j)} w_g^{(i)\prime}.$$

7. The asymptotic distribution of characteristic roots and vectors

The columns of the matrix C_n defined in section 2 are the characteristic vectors of D_n in the metric of A_n. We choose C_n so that the elements of the first row of $C_{ii}(n)$ are nonnegative. Let

$$(7.1) \qquad C_n = \begin{pmatrix} C_{11}(n) & C_{12}(n) & \ldots & C_{1,h+1}(n) \\ C_{21}(n) & C_{22}(n) & \ldots & C_{2,h+1}(n) \\ \vdots & \vdots & & \\ C_{h+1,1}(n) & C_{h+1,2}(n) & \ldots & C_{h+1,h+1}(n) \end{pmatrix}$$

$$= \begin{pmatrix} V_1(n) & 0 & \ldots & 0 \\ 0 & V_2(n) & \ldots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \ldots & V_{h+1}(n) \end{pmatrix} + \frac{1}{\sqrt{n}} \begin{pmatrix} Y_{11}(n) & Y_{12}(n) & \ldots & Y_{1,h+1}(n) \\ Y_{21}(n) & Y_{22}(n) & \ldots & Y_{2,h+1}(n) \\ \vdots & \vdots & & \\ Y_{h+1,1}(n) & Y_{h+1,2}(n) & \ldots & Y_{h+1,h+1}(n) \end{pmatrix}$$

$$= V_n + \frac{1}{\sqrt{n}} Y_n,$$

where

$$(7.2) \qquad V_i(n)V_i'(n) = I,$$

$$(7.3) \qquad V_i'(n)Y_{ii}(n) = Y_{ii}'(n)V_i(n).$$

Let

$$\bar{C}_n = X_n^{-1}. \tag{7.4}$$

If $X_n = J_n C_n^{-1}$, then $\bar{C}_n = C_n J_n$, where J_n is a diagonal matrix with diagonal elements $+1$ and -1. Now define $\bar{V}_n$ and $\bar{Y}_n$ in terms of $\bar{C}_n$ as V_n and Y_n are in terms of C_n. First let us show that for nonrandom $W_n \to W$, $Z_n \to Z$ (which define $X_n \to X$), $\bar{V}_n \to W'$ and $\bar{Y}_n \to -Z'$. We have

$$\begin{aligned}\bar{C}_n = \bar{V}_n + \frac{1}{\sqrt{n}}\bar{Y}_n &= \Big(W_n + \frac{1}{\sqrt{n}}Z_n\Big)^{-1} = \Big(I + \frac{1}{\sqrt{n}}W_n'Z_n\Big)^{-1}W_n' \\ &= \Big(I - \frac{1}{\sqrt{n}}W_n'Z_n + \frac{1}{n}(W_n'Z_n)^2 - \ldots\Big)W_n' \\ &= W_n' - \frac{1}{\sqrt{n}}W_n'Z_nW_n' + \frac{1}{n}(W_n'Z_n)^2W_n' - \ldots\end{aligned} \tag{7.5}$$

for n sufficiently large. The i-th diagonal block of (7.5) is

$$\begin{aligned}\bar{V}_i(n) + \frac{1}{\sqrt{n}}Y_{ii}(n) &= W_i'(n) - \frac{1}{\sqrt{n}}W_i'(n)Z_{ii}(n)W_i'(n) + \frac{1}{n}T_{ii}(n) \\ &= W_n'(n) - \frac{1}{\sqrt{n}}Z_{ii}'(n) + \frac{1}{n}T_{ii}(n),\end{aligned} \tag{7.6}$$

where the elements of $T_{ii}(n)$ are bounded. Multiplying each side of (7.6) on the left by its transpose, we obtain

$$\begin{aligned}I + \frac{1}{\sqrt{n}}[\bar{V}_i'(n)\bar{Y}_{ii}(n) + \bar{Y}_{ii}'(n)\bar{V}_i(n)] + \frac{1}{n}\bar{Y}_{ii}'(n)\bar{Y}_{ii}(n) \\ = I - \frac{1}{\sqrt{n}}[W_i(n)Z_{ii}'(n) + Z_{ii}(n)W_i'(n)] + \frac{1}{n}S_{ii}(n).\end{aligned} \tag{7.7}$$

Subtracting I from both sides, multiplying by $\bar{V}_i(n)$ and using (7.3) and (6.3), we find

$$Y_{ii}(n) = -\bar{V}_i(n)W_i(n)Z_{ii}'(n) + \frac{1}{\sqrt{n}}\bar{V}_i(n)R_i(n). \tag{7.8}$$

We can show (by means of an argument similar to that used in section 4) that the elements of $\bar{V}_i(n)R_i(n) = S_{ii}(n) - V_i(n)Y_{ii}'(n)Y_{ii}(n)$ and of $R_i(n)$ are bounded. Inserting (7.8) in (7.6) we obtain

$$\begin{aligned}\bar{V}_i(n) = \Big(W_i'(n) - \frac{1}{\sqrt{n}}Z_{ii}(n) + \frac{1}{n}T_{ii}(n)\Big)\Big(I + \frac{1}{\sqrt{n}}W_i(n)Z_{ii}'(n) \\ + \frac{1}{n}R_i(n)\Big)^{-1} = W_i'(n) + \frac{1}{n}Q_i(n)\end{aligned} \tag{7.9}$$

for n sufficiently large. It is clear from this that $\bar{V}_i(n) \to W_i'$ and from (7.8) that $\bar{Y}_{ii}(n) \to -Z_{ii}'$ (as $W_n \to W$ and $Z_n \to Z$). A nondiagonal block of (7.5) is

$$\frac{1}{\sqrt{n}}\bar{Y}_{ij}(n) = -\frac{1}{\sqrt{n}}W_i'(n)Z_{ij}(n)W_j'(n) + \frac{1}{n}T_{ij}(n). \tag{7.10}$$

Clearly, $\bar{Y}_{ij}(n) \to -W_i'Z_{ij}W_j'$.

$\bar{C}_n$ and C_n are different (for fixed W_n and Z_n) only in that columns of one are multiplied by -1 to obtain columns of the other. However, for n large enough the sign of the elements of the first row of $\bar{C}_{ii}(n)$ are those of $W_i'(n)$ $[= \bar{V}_i(n)]$, which are all positive (if the elements are different from 0). Thus $C_n = \bar{C}_n$ for n large enough. Therefore, for nonrandom $W_n \to W$ and $Z_n \to Z$, $V_n \to W'$, $Y_{ii}(n) \to -Z_{ii}'$ and $Y_{ij}(n) \to -W_i'Z_{ij}W_j'$. The discontinuities of the limiting transformation have limiting probability 0. Thus the limiting distribution of the random matrices $\Theta_n V_n$ and Y_n is the distribution of the random matrices Θ, $V = W'$, $Y = -W'ZW$ where the distribution of Θ, W and Z is given in theorem 5.

The distribution of Θ and $V' = W$ has been given explicitly. From the conditional distribution of Z let us find that of Y. Consider first Y_{ii}, $i \neq h+1$

$$\begin{aligned}(7.11) \qquad \mathcal{E}\{Y_{ii} \mid \Theta, W\} = -\mathcal{E}\{Z_{ii}' \mid \Theta, W\} &= \frac{1}{2(\lambda_i+2)} W_i'\Theta_i \\ &= \frac{1}{2(\lambda_i+2)} V_i\Theta_i .\end{aligned}$$

The covariance between two elements of Y_{ii}, say y_{ab} and $y_{a'b'}$ is

$$\frac{1}{2\lambda_i+4}(\delta_{aa'}\delta_{bb'} + v_{ab'}^{(i)} v_{a'b}^{(i)}) .$$

The matrix $Y_{h+1,h+1}$ is normal with mean $\mathbf{0}$ and covariance given above (for $\lambda_i = 0$).

Now consider $Y_{ij} = -W_i'Z_{ij}W_j'$ and $Y_{ji} = -W_j'Z_{ji}W_i'$. The joint conditional distribution is normal with zero means. The variance of the elements in Y_{ij} is $(\lambda_i + \lambda_j + \lambda_j^2)/(\lambda_i - \lambda_j)^2$ and that of the elements in Y_{ji} is $(\lambda_i + \lambda_j + \lambda_i^2)/(\lambda_i - \lambda_j)^2$. The covariance between elements in Y_{ij} is 0 as are those between elements in Y_{ji}. The covariance between an element y_{ab} in Y_{ij} and $y_{a'b'}$ in Y_{ji} is

$$-\frac{\lambda_i+\lambda_j+\lambda_i\lambda_j}{(\lambda_i-\lambda_j)^2} v_{ab'}^{(i)} v_{a'b}^{(j)} .$$

THEOREM 6. *Let*

$$D_n = \sum_{g=1}^{q_2} [y_g + \sqrt{n}\,\tau_g(n)\,\varepsilon_g][y_g + \sqrt{n}\,\tau_g(n)\,\varepsilon_g]' ,$$

where the p-dimensional vectors $y_1, \ldots, y_{q_2}$ $(q_2 \geqq p)$ *are independently distributed according to* $N(\mathbf{0}, I)$ *and* $\tau_g(n) = \sqrt{\lambda_i(n)} \to \sqrt{\lambda_i}$, $r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i$; $\lambda_i > \lambda_j$, $i < j$ $(i, j = 1, \ldots, h+1)$; $\lambda_{h+1}(n) = 0$. *Let* A_n *be independently distributed according to* $W(I, n)$. *Define* C_n *and* Φ_n *by means of* (2.11), (2.12) *and the conditions that* $c_{r_1+\ldots+r_{i-1}+1,g}(n) \geqq 0$, $r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i$ $(i = 1, \ldots, h+1)$ *and* Φ_n *is diagonal with diagonal elements in descending order. Let each of* C_n *and* Φ_n *be partitioned into* $(h+1)^2$ *submatrices of* $r_1, \ldots, r_{h+1}$ *rows and* $r_1, \ldots, r_{h+1}$ *columns. Let* Θ_n *be defined by* (6.4) *and* V_n *and* Y_n *be defined by* (7.1), (7.2) *and* (7.3). *The limiting distribution of* Θ_n, V_n *and* Y_n *is that of* Θ, V *and* Y *similarly partitioned, which can be described as follows: The matrices* Θ_i, V_i, Y_{ii} *are independent of* Θ_j, V_j, Y_{jj} $(i \neq j)$; *the distribution of* Θ_i *and* V_i' *is that of* Θ *and* W *given in theorem* 4 *with* $p = r_i$, $i \neq h+1$, *the conditional distribution of* Y_{ii} *given* Θ_i *and* V_i *is normal with mean* $(2\lambda_i + 4)^{-1}V_i\Theta_i$; *the distribution of* Θ_{h+1}, V_{h+1}' *is that of* Θ *and* W *given in theorem* 3 *with* $p = r_{h+1}$, $m = q_2 - p + r_{h+1}$; *the conditional distribution of* $Y_{h+1,\,h+1}$ *is normal with mean* 0; *the*

conditional distribution of Y *given* Θ *and* V *is normal; the conditional expectation of* Y_{ij}, $i \neq j$, *is* 0; *the conditional covariance matrix of* y_g *and* y_f', $r_1 + \ldots + r_{i-1} + 1 \leqq g, f \leqq r_1 + \ldots + r_i$, *consists of nondiagonal blocks of zeros and the i-th diagonal block is*

$$\frac{1}{2\lambda_i + 4}\left(I\delta_{gf} + v_f^{(i)} v_g^{(i)\prime}\right)$$

and the j-th diagonal block ($j \neq i$) *is*

$$\frac{\lambda_i + \lambda_j + \lambda_j^2}{(\lambda_i - \lambda_j)^2} I\delta_{gf};$$

the covariance matrix of y_g *and* y_f', $r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i$, $r_1 + \ldots + r_{j-1} + 1 \leqq f \leqq r_1 + \ldots + r_j$ ($i \neq j$) *consists of* 0's *except the j-th, i-th block which is*

$$\frac{\lambda_i + \lambda_j + \lambda_i\lambda_j}{(\lambda_i - \lambda_j)^2} v_f^{(j)} v_g^{(i)\prime}.$$

Now let us consider the asymptotic distribution of $C_n^* = \Gamma_n C_n$. Let Γ satisfy

$$\mathbf{B}_2 \lim_{n\to\infty} \bar{Q}_n \mathbf{B}_2' \Gamma = \Sigma\Gamma\Lambda. \tag{7.12}$$

If the diagonal elements of Λ are all different and if $\gamma_{1j} \neq 0, j = 1, \ldots, p$, then the restrictions $\gamma_{1j} > 0$, and $\gamma_j'\Sigma\gamma_j = 1$ determine Γ uniquely. If the same restrictions are placed on each Γ_n then $\Gamma_n \to \Gamma$ as $n \to \infty$ because the set of characteristic vectors is a continuous function of $\bar{Q}_n$ (which approaches the limit, $\lim_{n\to\infty} \bar{Q}_n$). If the diagonal elements of Λ are not all different, then another indeterminacy is involved. Partition Γ in the manner that the matrices in section 6 were partitioned;

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1,h+1} \\ \vdots & & \\ \Gamma_{h+1,1} & \cdots & \Gamma_{h+1,h+1} \end{pmatrix}. \tag{7.13}$$

Let O be an orthogonal matrix of the form

$$O = \begin{pmatrix} O_1 & 0 & \cdots & 0 \\ 0 & O_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & O_{h+1} \end{pmatrix}. \tag{7.14}$$

We require that

$$\Gamma'\Sigma\Gamma = I. \tag{7.15}$$

Then ΓO also satisfies (7.15); that is, there is an indeterminacy of such an orthogonal transformation. This indeterminacy can be removed by putting restrictions on Γ (such as requiring that the first column of Γ lie in a certain $r_1 - 1$ dimensional hyperplane, the second in a certain $r_1 - 2$ dimensional hyperplane, etc.,

with an element in each column having a specified sign). For all n greater than some particular integer, the same restrictions can be imposed on Γ_n. With these restrictions imposed, $\Gamma_n \to \Gamma$.

Then

$$C_n^* = \Gamma_n C_n = \Gamma_n V_n + \frac{1}{\sqrt{n}} \Gamma_n \bar{Y}_n . \tag{7.16}$$

Let

$$V_n^* = \Gamma_n V_n , \tag{7.17}$$

$$Y_n^* = \Gamma_n Y_n ; \tag{7.18}$$

then

$$C_n^* = V_n^* + \frac{1}{\sqrt{n}} Y_n^* . \tag{7.19}$$

V_n^* and Y_n^* are functions of V_n and Y_n. As $n \to \infty$, the functions approach limits; that is, for fixed $V_n = V$ and $Y_n = Y$, $V_n^* \to \Gamma V = V^*$, say, and $Y_n^* \to \Gamma Y = Y^*$, say. Thus by Rubin's theorem the limiting distribution of the random matrices V_n^* and Y_n^* is that of $\Gamma V = V^*$ and $\Gamma Y = Y^*$.

The distribution of V^* is that of ΓV. The distribution of Θ is given in section 6. The conditional distribution of Y^* given V^* and Θ can be found from that of Y. We have

$$\mathcal{E}\{Y_{ij}^* \mid \Theta, V\} = \sum_{k=1}^{h+1} \Gamma_{ik} \mathcal{E}\{Y_{kj} \mid \Theta, V\} = \Gamma_{ij} \frac{1}{2(\lambda_j + 2)} V_j \Theta_j , \qquad j \neq h+1 ; \tag{7.20}$$

$$\mathcal{E}\{Y_{i,h+1}^* \mid \Theta, V\} = 0 .$$

The conditional covariances are easily obtained from theorem 6. Let y_a^* be the a-th column of Y^*. Then the conditional covariance matrix of this vector for $a \leqq r_1$ is

$$\Gamma \begin{pmatrix} \frac{1}{2\lambda_1 + 4}(I + v_a^{(1)} v_a^{(1)\prime}) & 0 & \cdots & 0 \\ 0 & \frac{\lambda_1 + \lambda_2 + \lambda_2^2}{(\lambda_1 - \lambda_2)^2} I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{\lambda_1}{\lambda^2} I \end{pmatrix} \Gamma' \tag{7.21}$$

$$= \frac{1}{2\lambda_i + 4} \Gamma_1 (I + v_a^{(1)} v_a^{(1)\prime}) \Gamma_1' + \sum_{j=2}^{h+1} \frac{\lambda_i + \lambda_j + \lambda_j^2}{(\lambda_i - \lambda_j)^2} \Gamma_j \Gamma_j' ,$$

where

$$\Gamma_i = \begin{pmatrix} \Gamma_{1i} \\ \vdots \\ \Gamma_{h+1,i} \end{pmatrix} . \tag{7.22}$$

If $a \neq a^*$ and $a, a^* \leqq r_1$, the conditional covariances between y_a and y_{a^*} are

$$(7.23) \qquad \frac{1}{2\lambda_1+4}\,\Gamma_1 v_{a*}^{(1)} v_a^{(1)\prime}\Gamma_1' .$$

If $a \leqq r_1$ and $r_1 + 1 \leqq a^* \leqq r_1 + r_2$, then the covariances are

$$(7.24) \qquad \Gamma\begin{pmatrix} 0 & 0 & \dots & 0 \\ -\dfrac{\lambda_1+\lambda_2+\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\, v_{a*}^{(2)} v_a^{(1)\prime} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & 0 \end{pmatrix}\Gamma'$$

$$= -\frac{\lambda_1+\lambda_2+\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\,\Gamma_2 v_{a*}^{(2)} v_a^{(1)\prime}\Gamma_1' .$$

THEOREM 7. *Let the p-dimensional vectors* $x_\alpha(\alpha = 1, \dots N \geqq p)$ *be independently distributed according to* $N(\mathbf{B}_1 z_{1\alpha} + \mathbf{B}_2 z_{2\alpha}, \Sigma)$, *where the matrix of vectors* $(z_{i\alpha}', z_{2\alpha}')$ *is of rank p and* $z_{1\alpha}$ *has* q_1 *components and* $z_{2\alpha}$ *has* q_2 *components. Let* $B_n = [B_1(n), B_2(n)]$ *be defined by* (1.3); *let* $\bar{Q}_n = \frac{1}{n} Q_n$ *be defined by* (1.5) *where* $n = N - q_1 - q_2$; *let* A_n^* *be defined by* (1.6). *Let* Φ_n *be a diagonal matrix whose diagonal elements are the roots of* (1.7) *arranged in descending order of size. Let* $C_n^* = [c_1^*(n), \dots, c_p^*(n)]$ *be composed of the corresponding vectors satisfying* (1.8) *and* (1.9). *Let* $\tau_1^2(n)[\tau_1^2(n) \geqq \dots \geqq \tau_p^2(n)]$ *be the roots of* (2.1). *We assume* $\bar{Q}_n$ *approaches a non-singular limit in such a way that* $\tau_g(n) = \sqrt{\lambda_i(n)} \to \sqrt{\lambda_i}$, $r_1 + \dots + r_{i-1} + 1 \leqq g \leqq r_1 + \dots + r_i$, $\lambda_i \geqq \lambda_j$, $i < j$ $(i, j = 1, \dots, h+1)$, $\lambda_{h+1}(n) = 0$. *Let* Γ *be a matrix satisfying* (7.12) *and* (7.15) *where* Λ *is the limit of* (6.1), *and satisfying other restrictions to make* Γ *uniquely defined. Let* $\Gamma_n = [\gamma_1(n), \dots, \gamma_p(n)]$ *be composed of vectors satisfying* (2.2) *and* (2.3) *and the additional restrictions on* Γ *(for n sufficiently large). Let* Θ_n *be defined by* (6.4). *Let* $C_n^* = V_n^* + \frac{1}{\sqrt{n}} Y_n^*$, *where*

$$(7.25) \qquad \Gamma_n^{-1} V_n^* = V_n = \begin{pmatrix} V_1(n) & 0 & \dots & 0 \\ 0 & V_2(n) & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & V_{h+1}(n) \end{pmatrix},$$

$V_i'(n)V_i(n) = I$, *the elements of the first row of* $V_i(n)$ *are nonnegative and* $V_i'(n)Y_{ii}(n) = Y_{ii}'(n)V_i(n)$, *where* $Y_{ii}(n)$ *is the i-th diagonal submatrix of* $\Gamma_n^{-1}Y_n^* = Y_n$. *As* $n \to \infty$, *the limiting distribution of* Θ_n, V_n^* *and* Y_n^* *is that of* Θ, ΓV *and* Y^*, *similarly partitioned, which may be described as follows: The marginal distribution of* Θ *and V is such that* Θ_i, V_i *is independent of* Θ_j, V_j $(i \neq j)$; *the marginal distribution of* Θ_i, V_i' *is that of* Θ, *W given in theorem* 4 *with* $p = r_i$, $i \neq h+1$; *the distribution of* Θ_{h+1}, V_{h+1}' *is that of* Θ, *W given in theorem* 3 *with* $p = r_{h+1}$, $m = q_2 - p + r_{h+1}$; *the conditional distribution of* Y^* *given* Θ, *V is normal; the conditional expectation of*

a submatrix Y_{ij}^* *is given by* (7.19); *the conditional covariance matrix of* y_g^* *and* $y_f^{*\prime}$, $r_1 + \ldots + r_{i-1} + 1 \leqq g, f \leqq r_1 + \ldots + r_1$, *is*

$$(7.26) \qquad \frac{1}{2\lambda_i + 4}[\Gamma_i v_f^{(i)} v_g^{(i)\prime} \Gamma_i' + \delta_{gf} \Gamma_i \Gamma_i'] + \delta_{gf} \sum_{\substack{j=1 \\ j \neq i}}^{h+1} \frac{\lambda_i + \lambda_j + \lambda_j^2}{(\lambda_i - \lambda_j)^2} \Gamma_j \Gamma_j';$$

the conditional covariance between y_g^* *and* $y_f^{*\prime}$, $r_1 + \ldots + r_{i-1} + 1 \leqq g \leqq r_1 + \ldots + r_i, r_1 + \ldots + r_{j-1} + 1 \leqq f \leqq r_1 + \ldots + r_j$ $(i \neq j)$, *is*

$$(7.27) \qquad -\frac{\lambda_i + \lambda_j + \lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \Gamma_j v_f^{(j)} v_g^{(i)\prime} \Gamma_i$$

where Γ_i *is defined in* (7.22) *and* $v_g^{(i)}$ *is the g-th column of* V_i.

A special case of considerable interest is the case of $r_i = 1$, $i \neq h + 1$ and $r_{h+1} = p - h$. Then Y_{ij} consists of one element for $i, j \neq h + 1$. In this case $V_i = 1$ for $i \neq h + 1$. We can easily express the conditional distribution of Y given V_{h+1} by integrating out $\theta_1, \ldots, \theta_h$. The marginal distribution of θ_i is $N(0, 2\lambda_i^2 + 4\lambda_i)$ and the conditional distribution of y_{ii} is $N[\theta_i/(2\lambda_i + 4), 1/(\lambda_i + 2)]$. θ_i and y_{ii} are independent of the other variables. The marginal distribution then of y_{ii} is $N(0, 1/2)$.

In this case $\mathcal{E}\{Y^* | V_{h+1}\} = 0$. The conditional covariance between y_a^* and $y_a^{*\prime}$, $a \leqq h$ is

$$(7.28) \qquad \mathcal{E} y_a^* y_a^{*\prime} = \tfrac{1}{2}\Gamma_a \Gamma_a' + \sum_{\substack{j=1 \\ j \neq a}}^{h+1} \frac{\lambda_a + \lambda_j + \lambda_j^2}{(\lambda_a - \lambda_j)^2} \Gamma_j \Gamma_j'.$$

The conditional covariance between y_a^* and $y_{a^*}^*$, $a \neq a^*$, $a, a^* \leqq h$ is

$$(7.29) \qquad \mathcal{E}\{y_a^* y_{a^*}^{*\prime} | V_{h+1}\} = -\frac{\lambda_a + \lambda_{a^*} + \lambda_a \lambda_{a^*}}{(\lambda_a - \lambda_{a^*})^2} \Gamma_{a^*} \Gamma_a'.$$

The conditional covariance between y_a^* and $y_{a^*}^*$, $a \leqq h$, $h + 1 \leqq a^* \leqq p$, is

$$(7.30) \qquad \mathcal{E}\{y_a^* y_{a^*}^{*\prime} | V_{h+1}\} = \frac{1}{\lambda_a} \Gamma_{h+1} v_{a^*}^{(h+1)} \Gamma_a'.$$

The conditional covariance between y_a^* and $y_{a^*}^{*\prime}$, $h + 1 \leqq a, a^* \leqq p$, is

$$(7.31) \quad \mathcal{E}\{y_a^* y_{a^*}^{*\prime} | V_{h+1}\} = \tfrac{1}{4}\Gamma_{h+1}(I\delta_{aa^*} + v_{a^*}^{(h+1)} v_a^{(h+1)\prime})\Gamma_{h+1}' + \sum_{j=1}^{h} \frac{1}{\lambda_j} \Gamma_j \Gamma_j' \delta_{aa^*}.$$

Clearly, if $r_i = 1$, $i = 1, \ldots, h + 1 = p$, then $V_n = I$, and the limiting distribution of Θ_n and Y_n^* is such that the marginal distribution of Y^* is normal with mean 0 and covariances derived from above.

8. Remarks

8.1. *Use of N and* $n = N - q$. In section 2 we defined $\overline{Q}_n$ as $\frac{1}{n} Q_n$. The asymptotic distributions obtained are exactly the same if one uses $\frac{1}{N} Q_n$ for the roots of $|B_2 \frac{1}{N} Q_n B_2' - \rho^2 \Sigma|$ are multiples by n/N of the roots of (2.1). These roots converge also to $\lambda_1, \ldots, \lambda_{h+1}$, respectively, and for each n the multiplicities are the same in the two cases. Using $\frac{1}{N} Q_n$ changes the definition of the sample roots again

by a factor of n/N. We might also define A^* in terms of N instead of n and normalized c_g^* in terms of N instead of n. Asymptotically the effect of using N instead of n disappears. Rubin's theorem can be used to prove each such statement rigorously.

8.2. *The limiting probabilities.* It is interesting to see for what sequences of sets in the space of the characteristic vectors the limiting probabilities are defined. As a simple example, suppose $p = r_1 = 2$ and $\Sigma = I$. We shall consider a sequence of sets for one vector c_1 and another sequence for c_2 defined in the same plane. Consider a segment on the unit circle in the right half plane. The regions for c_1 contain this segment and as $n \to \infty$ the regions converge to the segment. The boundaries converge as $1/\sqrt{n}$. Consider the segment of the unit circle in the right half plane composed of the points which are 90° from the points in the other segment. There is a corresponding sequence of regions which close down on this segment as n increases. The limiting probabilities are defined for these sequences.

8.3. *Cases of special interest.* Two cases of the model discussed here are of particular interest. The one occurs when the "fixed variate" vectors z_α (in section 1) are composed of dummy variates; that is, variates that are 0 or 1 (see [1], for example). These can be chosen so that $\mathbf{B}z_\alpha = \mu_i$, $\alpha = N_1 + \ldots + N_{i-1} + 1, \ldots, N_1 + \ldots + N_i$ $(N_1 + \ldots + N_q = N)$. The first N_1 x_α are observations from the first population, etc. If we require that $N_i = k_i N$ as $N \to \infty$, then the multiplicities of the roots of (2.1) are unchanged as $N \to \infty$.

It can be shown that if $\sqrt{n}[\tau_g^2(n) - \tau_f^2(n)] \to 0$ as $n \to \infty$, $r_1 + \ldots + r_{i-1} + 1 \leqq g, f \leqq r_1 + \ldots + r_i$, $i = 1, \ldots, h+1$, then Θ_n, W_n, and Z_n (also Θ_n, V_n^*, and Y_n^*) can be defined in the manner of this paper and have the same limiting distribution as given here. Thus we can weaken our conditions slightly. If $N_i = k_i N$ to within rounding error in the case mentioned above, the same theory applies.

If the fixed variate subvectors $z_{2\alpha}$ are not composed of dummy variates, in general the nonzero roots of (2.1) will be simple for all n and in the limit. Then the theory at the end of section 7 applies.

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